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Improved Lindstedt–Poincaré method for the solution of nonlinear problems

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Abstract

We apply the Linear Delta Expansion (LDE) to the Lindstedt–Poincaré (“distorted time”) method to find improved approximate solutions to nonlinear problems. We find that our method works very well for a wide range of parameters in the case of the anharmonic oscillator (Duffing equation), of the non linear pendulum and of more general anharmonic potentials. The approximate solutions found with this method converge more rapidly to the exact ones than in the simple Lindstedt–Poincaré method.

1. Introduction

The study of nonlinear problems is of crucial importance in all areas of Physics. Some of the most interesting features of physical systems are hidden in their nonlinear behavior and must be studied with appropriate methods designed to tackle nonlinear problems. In general, given the nature of nonlinear phenomena, the approximation methods can only be applied within certain ranges of the physical parameters and or to certain classes of problems. It is a challenge to devise nonlinear frameworks that contain both operational ease and flexibility in their application. In this paper we present a method for the solution of nonlinear problems that attempts to accomplish these features.

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There are several methods which have been used to find approximate solutions to nonlinear problems; here, we just review a few. Lindstedt developed a method a long time ago [1] in which one considers solutions to problems involving conservative oscillatory systems with an unknown period. The main observation is that by introducing a rescaled time, one can avoid the appearance of terms indefinitely growing with time (“secular terms”), that are common in ordinary perturbation theory. The method is now known as the Lindstedt–Poincaré (LP) method or as the Distorted Time method.

Another known technique is the perturbative δ expansion (see for example Ref. [2]). In this case the idea is to modify the exponent of the nonlinear term by introducing a parameter δ as new exponent. δ interpolates between the linear ($\delta = 0$) and the nonlinear ($\delta = 1$) problems. If one is able to solve the linear problem then the original nonlinear problem becomes, after a power expansion in δ, an infinite sequence of linear problems which are (formally) solvable.

Yet another framework is the Multiple-Scale Perturbation Theory (MSPT) [3]. In this case, one tackles problems in which a dynamical system has physical behaviors at various length or time scales. This is usually problematic for ordinary perturbation theory due to the appearance (again) of secular terms. The central idea is to introduce more than one time and to treat them as independent variables. By performing the usual perturbative expansion, one then imposes conditions on the solutions (which depend on the different “times”) in order to get rid of secular terms and a linear differential equation is left to solve.

Finally, another method is the Linear Delta Expansion (LDE) [4]. This is a method in which an arbitrary (or several) parameter λ is introduced into the problem and calculations are carried out with conventional perturbation theory in an expansion parameter $\delta = 1$. At each order in δ, the convergence of the approximation can be improved by applying the principal of minimal sensitivity which consists of a minimization of an observable with respect to the parameter λ.

All of these methods have been applied to a variety of problems. In Ref. [2], Bender et al. showed how one can obtain approximate solutions using the perturbative δ expansion and the MSPT to the Duffing equation (the classical anharmonic oscillator). Its success then has motivated their extension of the method into quantum systems [3]. The LDE method has extensively been applied in many different settings with varying degrees of success. For example, in Ref. [5] it has been used to analyze disordered systems; in Ref. [6] it has been applied to study the slow roll potential in inflationary models. Pinto and collaborators have applied it to the Bose–Einstein condensation problem [7], the $O(N)(\phi^2)^2_{3d}$ model [8], to the Walecka model [9] and to the ϕ^4 theory at high temperature [10]. Detailed references can be found in these works.

We can see that it is possible to tackle a large number of nonlinear problems with these well known techniques. However, there is still room for substantial improvement over them. As mentioned before, it is desirable to have a method that works over a large range of parameters, which is not always the case in the aforementioned methods, and we would like the new method to give a smaller error in the approximations than its competitors. It is also desirable to devise a framework with operational flexibility and easy to adapt to many different problems.

We show that the method presented in this paper accomplishes these goals in the case of the Duffing equation, of more general anharmonic potentials and of the nonlinear pendulum. The method is based on the application of the LDE to the LP method [11]. We find solutions that converge much faster than in the other methods described.

In Section 2 a brief review of the LP method is presented followed by a review of the LDE method in Section 3. We then show the application of both methods to three problems: to the anharmonic oscillator in Section 4, to more general anharmonic potentials in Section 5 and to the nonlinear pendulum in Section 6. We present our conclusions and current work in Section 7. Appendix A contains some of the formulae employed in the computations.

2. The Lindstedt–Poincaré method

In this section we introduce the Lindstedt–Poincaré distorted time (LP) method [1]. We consider a nonlinear ODE of the form

$$\ddot{x}(t) + \omega^2 x(t) = \varepsilon f(x(t)), \tag{1}$$

which describes a conservative system, oscillating with an unknown period T. The nonlinear term $\varepsilon f(x(t))$ is treated as a perturbation. Unfortunately, when the ordinary perturbation is applied to Eq. (1), by writing the solution as a series in ε, the appearance of secular terms spoils the expansion and any predictive power is lost for sufficiently large time scales.

In order to avoid the appearance of secular terms, we switch to a scaled time $\tau = 2\pi t/T \equiv \Omega t$, where T is the (unknown) period of the oscillations. The ODE now reads

$$\Omega^2 \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}(\tau) + \omega^2 x(\tau) = \varepsilon f(x(\tau)). \tag{2}$$

We notice that the dependence upon ε in this equation enters both in the solution $x(\tau)$ and in the frequency Ω. By assuming ε to be a small parameter we write

$$\Omega^2 = \sum_{n=0}^{\infty} \varepsilon^n \alpha_n; \quad x(\tau) = \sum_{n=0}^{\infty} \varepsilon^n x_n(\tau)$$

and expand the r.h.s of Eq. (2) as

$$f(x) = f\left(\sum_{n=0}^{\infty} \varepsilon^n x_n(\tau)\right) \approx f(x_0) + \varepsilon x_1 f'(x_0) + \varepsilon^2 \left[x_2 f'(x_0) + \frac{x_1^2}{2} f''(x_0)\right] + \varepsilon^3 \left[x_3 f'(x_0) + x_2 x_1 f''(x_0) + \frac{x_1^3}{6} f'''(x_0)\right] + O[\varepsilon^4].$$

By using these expansions inside Eq. (2) we obtain a system of linear inhomogeneous differential equations, each corresponding to a different order in ε. Let us consider the first few terms. To order ε^0 we obtain the equation

$$\alpha_0 \frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} + \omega^2 x_0(\tau) = 0, \tag{3}$$

describing a harmonic oscillator of frequency $\Omega = \sqrt{\alpha_0} = \omega$. To order ε we obtain the equation

$$\alpha_0 \frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + \omega^2 x_1(\tau) = s_1(\tau), \tag{4}$$

where the r.h.s. is given by

$$s_1(\tau) \equiv -\alpha_1 \frac{d^2 x_0}{d\tau^2} + f(x_0). \tag{5}$$

We stress the oscillatory behavior of the driving term $s_1(\tau)$, because of its dependence upon the order-0 solution, $x_0(\tau)$. As a result $s_1(\tau)$ will contain the fundamental frequency, corresponding to a period of 2π in the scaled time, and multiples of this frequency, appearing through the term $f(x_0(\tau))$. The presence of a driving term with the fundamental frequency leads to a resonant behavior of $x_1(\tau)$ and to the unfortunate occurrence of secular terms, which spoils our expansion. However, we can deal with this problem by fixing the coefficient α_1 to cancel the resonant term in the r.h.s. of Eq. (4). The iteration of this procedure to a given order n allows to determine the coefficients $\alpha_0, \ldots, \alpha_n$ and therefore the frequency $\Omega = \sqrt{\alpha_0 + \varepsilon\alpha_1 + \cdots + \varepsilon^n \alpha_n}$.

3. Linear delta expansion

The linear delta expansion (LDE) is a powerful technique that was originally introduced to deal with problems of strong coupling Quantum Field Theory, for which the naive perturbative approach is not useful. Since then this method has been applied to a wide class of problems [5–10]. In its original formulation a Lagrangian density $\mathscr{L}$, which is not exactly solvable, is interpolated with a solvable Lagrangian $\mathscr{L}_0(\lambda)$, depending upon one (or more) parameters λ:

$$\mathscr{L}_\delta = \mathscr{L}_0(\lambda) + \delta(\mathscr{L} - \mathscr{L}_0(\lambda)). \tag{6}$$

For $\delta = 0$ one obtains $\mathscr{L}_0(\lambda)$, whereas for $\delta = 1$ one recovers the full Lagrangian $\mathscr{L}$. The term $\delta(\mathscr{L} - \mathscr{L}_0)$ is treated as a perturbation and δ is used to keep track of the perturbative order. We stress that δ is not a "natural" expansion parameter, as in ordinary perturbation theory, since it is not present in the original theory. Eventually δ is set to be 1.

We notice that the interpolation of the full Lagrangian with the solvable one, $\mathscr{L}_0(\lambda)$, brings an artificial dependence upon the arbitrary parameter λ. Such dependence, which would vanish if all perturbative orders were calculated, can be eliminated to a finite perturbative order, by requiring some physical observable $\mathcal{O}$ to be locally insensitive to λ, i.e.

$$\frac{\partial \mathcal{O}(\lambda)}{\partial \lambda} = 0.$$

This condition is known as Principle of Minimal Sensitivity (PMS) and is normally seen to improve the convergence to the exact solution. Rigorous proofs of convergence of the LDE applied to quantum problems have been obtained in Ref. [12].

4. Anharmonic oscillator

In this section, we apply the LDE to the LP method in order to find approximate solutions to the Duffing equation, a problem which has already been considered in Ref. [11]; here, we present the calculation in more detail.

Consider the equation for the anharmonic oscillator,

$$\frac{d^2x}{dt^2}(t) + \omega^2 x(t) = -\mu x^3(t). \tag{7}$$

This equation describes a conservative system, where the total energy is given by

$$E = \frac{\dot{x}^2}{2} + \left[\frac{\omega^2 x^2}{2} + \mu\frac{x^4}{4}\right]. \tag{8}$$

The period of the oscillation can be calculated in terms of an elliptic integral,

$$T_{\text{exact}} = 2\int_{-A}^{A} dx \frac{1}{\sqrt{2(E - V(x))}}, \tag{9}$$

where A is the amplitude of the oscillations.

Following the procedure explained in the Sections 2 and 3, we write Eq. (7) as

$$\Omega^2 \frac{d^2x}{d\tau^2}(\tau) + (\omega^2 + \lambda^2)x(\tau) = \delta[-\mu x^3(\tau) + \lambda^2 x(\tau)], \tag{10}$$

where an arbitrary parameter λ with dimension of frequency has been introduced. Clearly for $\delta = 1$, Eq. (10) reduces to Eq. (7). We repeat the procedures previously explained and find a hierarchy of linear inhomogeneous differential equations toto be solved sequentially.

4.1. Zeroth order

To zeroth order we obtain the equation

$$\alpha_0 \frac{d^2x_0}{d\tau^2} + (\omega^2 + \lambda^2)x_0(\tau) = 0, \tag{11}$$

with solution

$$x_0(\tau) = A\cos\tau. \tag{12}$$

The zeroth-order frequency is then given by

$$\alpha_0 = \omega^2 + \lambda^2. \tag{13}$$

4.2. First order

To first order we find the equation

$$\alpha_0 \frac{d^2x_1}{d\tau^2} + (\omega^2 + \lambda^2)x_1(\tau) = S_1(\tau), \tag{14}$$

where

$$S_1(\tau) = A\cos\tau\left[\alpha_1 + \lambda^2 - \frac{3A^2\mu}{4}\right] - \frac{A^3\mu}{4}\cos 3\tau. \tag{15}$$

Now α_1 is fixed by eliminating the term proportional to $\cos\tau$:

$$\alpha_1 = \frac{3A^2\mu}{4} - \lambda^2. \tag{16}$$

We obtain the solution

$$x_1(\tau) = -\frac{A^3\mu}{32(\omega^2+\lambda^2)}\cos\tau + \frac{A^3\mu}{32(\omega^2+\lambda^2)}\cos 3\tau,$$

and the frequency

$$\Omega^2 = \alpha_0 + \alpha_1 = \omega^2 + \frac{3A^2\mu}{4}, \tag{17}$$

which is observed to be independent of λ.

4.3. Second order

The second-order equation is given by

$$\alpha_0 \frac{d^2x_2}{d\tau^2} + (\omega^2+\lambda^2)x_2(\tau) = S_2(\tau), \tag{18}$$

where now

$$S_2(\tau) = \frac{A(3A^4\mu^2 + 128\alpha_2(\omega^2+\lambda^2))}{128(\omega^2+\lambda^2)}\cos\tau + \frac{A^3\mu(3A^2\mu - 4\lambda^2)}{16(\omega^2+\lambda^2)}\cos 3\tau - \frac{3A^5\mu^2}{128(\omega^2+\lambda^2)}\cos 5\tau. \tag{19}$$

As before α_2 is fixed by eliminating the term proportional to $\cos\tau$:

$$\alpha_2 = -\frac{3A^4\mu^2}{128(\omega^2+\lambda^2)}. \tag{20}$$

We obtain the solution

$$x_2(\tau) = \frac{A^3\mu(23A^2\mu - 32\lambda^2)}{1024(\omega^2+\lambda^2)^2}\cos\tau + \frac{A^3\mu(-3A^2\mu + 4\lambda^2)}{128(\omega^2+\lambda^2)^2}\cos 3\tau + \frac{A^5\mu^2}{1024(\omega^2+\lambda^2)^2}\cos 5\tau \tag{21}$$

and the frequency

$$\Omega^2 = \alpha_0 + \alpha_1 + \alpha_2 = \omega^2 + \frac{3A^2\mu}{4} - \frac{3A^4\mu^2}{128(\omega^2+\lambda^2)}. \tag{22}$$

Note that at this order the frequency now depends on the arbitrary parameter λ. However, due to the explicit dependence, by applying the PMS, we would obtain the same solution as in the simple LP method ($\lambda = 0$). In order to get a different solution, we must go to the next order in the expansion.

4.4. Third order

Following the same procedure, we obtain the following expression for the third order:

$$\alpha_0 \frac{d^2 x_3}{d\tau^2} + (\omega^2 + \lambda^2) x_3(\tau) = S_3(\tau), \tag{23}$$

where

$$s_3(\tau) = \left[A\alpha_3 - \frac{3A^5\mu^2(3A^2\mu - 4\lambda^2)}{512(\omega^2+\lambda^2)^2} \right] \cos\tau - \frac{(A^3\mu(297A^4\mu^2 - 768A^2\mu\lambda^2 + 512\lambda^4))}{2048(\omega^2+\lambda^2)^2} \cos 3\tau$$
$$+ \frac{3A^5\mu^2(3A^2\mu - 4\lambda^2)}{256(\lambda^2+\omega^2)^2} \cos 5\tau - \frac{3A^7\mu^3}{2048(\lambda^2+\omega^2)^2} \cos 7\tau. \tag{24}$$

By eliminating the term proportional to $\cos\tau$ we determine α_3 to be

$$\alpha_3 = \frac{3A^4\mu^2(3A^2\mu - 4\lambda^2)}{512(\lambda^2+\omega^2)^2}, \tag{25}$$

and the solution

$$\begin{aligned} x_3(\tau) = & -\frac{A^3\mu}{32768} \frac{547A^4\mu^2 - 1472A^2\mu\lambda^2 + 1024\lambda^4}{(\lambda^2+\omega^2)^3} \cos\tau \\ & + \frac{A^3\mu}{16384} \frac{297A^4\mu^2 - 768A^2\mu\lambda^2 + 512\lambda^4}{(\lambda^2+\omega^2)^3} \cos 3\tau \\ & + \frac{A^5\mu^2}{2048} \frac{(-3A^2\mu + 4\lambda^2)}{(\lambda^2+\omega^2)^3} \cos 5\tau + \frac{A^7\mu^3}{32768} \frac{1}{(\lambda^2+\omega^2)^3} \cos 7\tau. \end{aligned}$$

The frequency to order δ^3 is now obtained to be

$$\Omega^2 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \omega^2 + \frac{3A^2\mu}{4} - \frac{3A^4\mu^2}{128(\omega^2+\lambda^2)} + \frac{3A^4\mu^2(3A^2\mu - 4\lambda^2)}{512(\lambda^2+\omega^2)^2}. \tag{26}$$

This time, the frequency depends upon the arbitrary parameter λ in a nontrivial way and we can apply the PMS in order to fix the value of λ. We do this by imposing that $d\Omega^2/d\lambda = 0$, which leads to the following result:

$$\lambda^2 = \frac{3A^2\mu}{4}. \tag{27}$$

Notice that since λ depends linearly upon A the formula for Ω^2 obtained in this case *does not simply correspond to an expansion in A*. As a matter of fact we find that the frequency corresponding to this value of λ is

$$\Omega^2 = \frac{69A^4\mu^2 + 192A^2\mu\omega^2 + 128\omega^4}{96A^2\mu + 128\omega^2}. \tag{28}$$

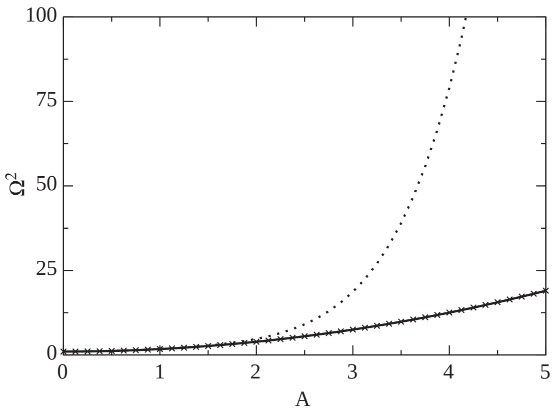

Fig. 1. Squared frequency of the anharmonic oscillator as a function of the amplitude (arbitrary units). We assume $\omega = \mu = 1$. The solid curve is the exact result, the dotted curve is the Lindstedt–Poincaré result and the dash-cross curve is the result of our method.

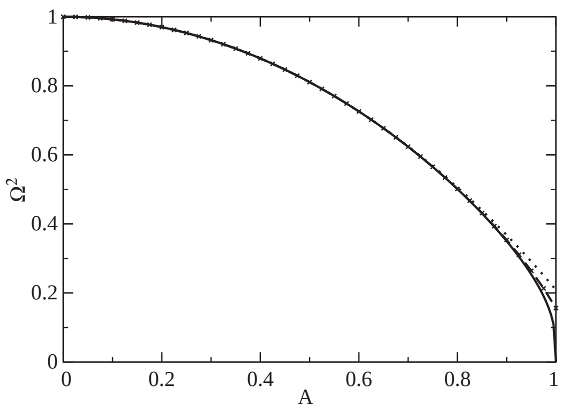

Fig. 2. Squared frequency of the anharmonic oscillator as a function of the amplitude (arbitrary units). We assume $\omega = 1$ and $\mu = -1$. The solid curve is the exact result, the dotted curve is the Lindstedt–Poincaré result and the dash-cross curve is the result of our method.

Note that the Duffing equation (7) is left invariant under the simultaneous rescaling of the anharmonic coupling μ and of the amplitude, i.e. $\mu \to \mu'$ and $A \to A' = A\sqrt{\mu/\mu'}$. This invariance is manifest in Eq. (28), which is function of $A^2\mu$, which is invariant under this rescaling.

In Fig. 1 we compare the exact frequency, calculated with Eq. (9) with the frequency obtained with our method (LPLDE), Eq. (28), and with the LP method, Eq. (26), taking $\lambda = 0$, both to third order in perturbation theory. We take $\omega = \mu = 1$ (see the left plot of Fig. 3) and vary the amplitude of the oscillations. We observe that our method yields an excellent approximation to the exact result even for large amplitudes, where the simple LP approximation fails.

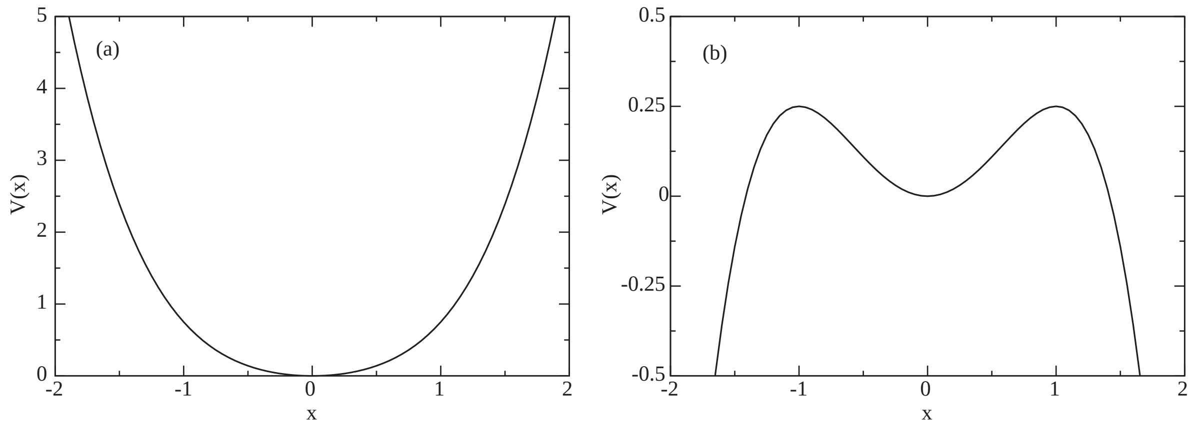

Fig. 3. Anharmonic potential corresponding to (a) $\omega = \mu = 1$ and (b) $\omega = 1$ and $\mu = -1$.

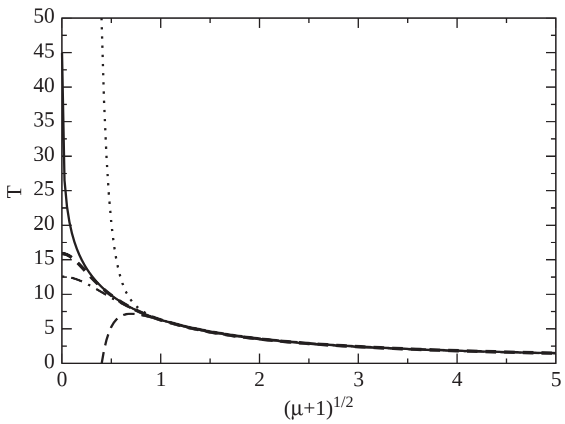

Fig. 4. Period of the anharmonic oscillator. We assume $\omega = 1$. The solid curve is the exact result; the dashed and dotted curve refer to the formulas of Ref. [2] calculated to first- and second-order respectively; the dot-dashed curve is the Lindstedt–Poincaré result; the bold dashed curve is the result of our method.

In Fig. 2 we consider the case studied in Fig. 1, but choosing $\omega = 1$ and $\mu = -1$ (see the right plot of Fig. 3). In this case the potential has a local minimum in the origin and two maxima, located at $x = \pm 1$. Periodic solutions are supported only for amplitudes $A < 1$, $A = 1$ being a point of (unstable) equilibrium, where the period diverges. Also in this case, the LPLDE method offers an excellent approximation to the exact result for a large range of amplitudes; as expected, the approximation is poorer in the region $A \approx 1$, where the point of equilibrium is approached.

In Fig. 4 we compare the period obtained with our method to the exact period of Eq. (9) and to the one obtained with the formulae of [2], which are obtained by applying the nonlinear delta expansion. Our method provides an excellent approximation to the exact period over a wide range of the parameter μ, which controls the nonlinearity. The plots are obtained assuming $\omega = 1$ and the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. The formulae of Ref. [2] behave badly in the region $\mu < 0$, which corresponds to a potential well of finite depth centered around $x = 0$, and yield a

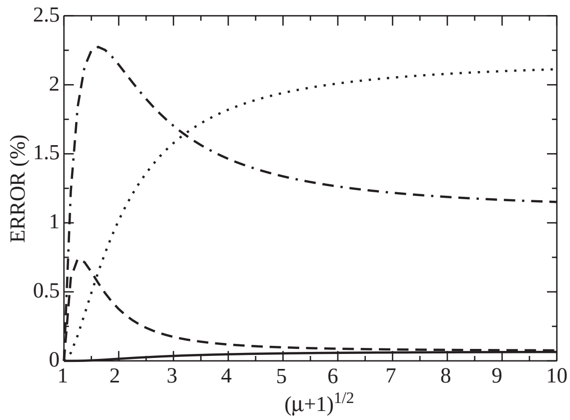

Fig. 5. Error corresponding to the different approaches for the case studied in Fig. 4. We assume $\omega = 1$. The error is defined as $\Delta \equiv (\Omega^2_{\text{approx}} - \Omega^2_{\text{exact}})/\Omega^2_{\text{exact}} \times 100$. The dot-dashed and dashed curves refer to the formulas of Ref. [2] calculated to first and second order respectively; the dotted curve is the Lindstedt–Poincaré result; the solid curve is the result of our method.

precision comparable to the one achieved with our method for $\mu > 0$. Corresponding to the value $\mu = 0$ the oscillator is in a position of (unstable) equilibrium and the exact period diverges. Notice that for large values of μ all the methods seem to give a good approximation to the exact solution, including the LP method (to first order), which (to third order) was behaving poorly in the case previously studied. Unfortunately the equations of Ref. [2] are not suitable to be analyzed as in Fig. 1, and thus we cannot fully test the efficiency of this method.

In Fig. 5, we plot the relative error over the squared frequency corresponding to the different approximations for $\mu > 0$ (the error is given by $\Delta = (\Omega^2_{\text{approx}} - \Omega^2_{\text{exact}})/\Omega^2_{\text{exact}} \times 100$). Our method to third order in perturbation theory yields an error typically smaller than the errors of the other methods and with a magnitude of about 0.1%.

5. More general anharmonic potentials

In a more general case than the one discussed in Section 4, we consider a unit mass under the action of a force $f(x)$. The motion is described by Newton's equation

$$\frac{d^2x}{dt^2} = f(x), \tag{29}$$

where $f(x)$ can be a nonlinear function of the position. In the proximity of a stable equilibrium point it will be possible to approximate $f(x)$ with the first few terms in the Taylor expansion. To make things easier we limit ourselves to the case in which $f(x)$ is an odd function of x and write the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = -a_3x^3 - a_5x^5 - a_7x^7 - \cdots, \tag{30}$$

where a_i are constants.

The application of our method to this equation where the coefficients a_i are assumed to vanish for $i>7$ is straightforward and follows closely the procedure illustrated in the previous section. In particular, to third order we obtain

$$\begin{aligned}\Omega^2 &= [\lambda^2+\omega^2]+\delta\left[\frac{3a_3A^2}{4}+\frac{5a_5A^4}{8}+\frac{35a_7A^6}{64}-\lambda^2\right]+\delta^2\frac{A^4}{196608(\omega^2+\lambda^2)}\left[-4608a_3^2-12288a_3a_5A^2\right.\\ &\quad\left.-32(260a_5^2+423a_3a_7)A^4-18528a_5a_7A^6-10395a_7^2A\right]+\delta^3\frac{1}{301989888(\lambda^2+\omega^2)^2}\\ &\quad\times\left[5308416a_3^3A^6+18948096a_3^2a_5A^8+22364160a_3a_5^2A^{10}+20155392a_3^2a_7A^{10}\right.\\ &\quad+8642560a_5^3A^{12}+47609856a_3a_5a_7A^{12}+27479040a_5^2a_7A^{14}+25418880a_3a_7^2A^{14}\\ &\quad+29136600a_5a_7^2A^{16}+10286325a_7^3A^{18}-7077888a_3^2A^4\lambda^2-18874368a_3a_5A^6\lambda^2\\ &\quad\left.-12779520a_5^2A^8\lambda^2-20791296a_3a_7A^8\lambda^2-28459008a_5a_7A^{10}\lambda^2-15966720a_7^2A^{12}\lambda^2\right].\end{aligned}\tag{31}$$

The application of the PMS, after setting $\delta=1$, in this case yields

$$\lambda_{\mathrm{PMS}}=\sqrt{\frac{\mathcal{N}}{\mathcal{D}}},\tag{32}$$

where

$$\begin{aligned}\mathcal{N} &= 6(5308416a_3^3A^2+18948096a_3^2a_5A^4+3072a_3(7280a_5^2+6561a_3a_7)A^6\\ &\quad+2048a_5(4220a_5^2+23247a_3a_7)A^8+1920a_7(14312a_5^2+13239a_3a_7)A^{10}\\ &\quad+29136600a_5a_7^2A^{12}+10286325a_7^3A^{14}),\end{aligned}\tag{33}$$

$$\mathcal{D}=9216(4608a_3^2+12288a_3a_5A^2+32(260a_5^2+423a_3a_7)A^4+18528a_5a_7A^6+10395a_7^2A^8).\tag{34}$$

Notice that Eqs. (31) and (32) reduce to Eqs. (26) and (27) for $a_5=a_7=0$ and $a_3=\mu$.

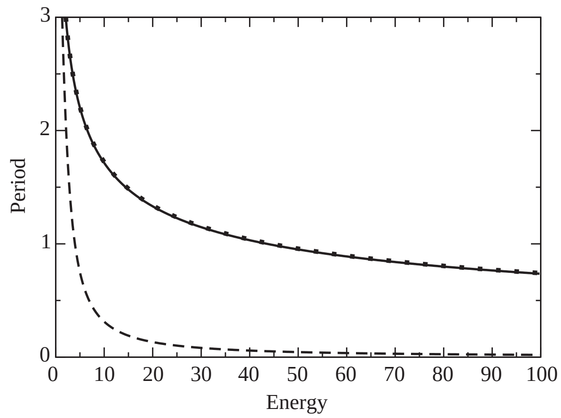

Fig. 6. Period of the solution of Eq. (30) as a function of the energy by assuming $\omega=a_3=a_5=a_7=1$. The solid line is the period obtained with our method, the dotted line is the exact result and the dashed line is the Lindstedt–Poincaré result.

In Fig. 6 we have applied the formulas given above, by choosing $\omega = a_3 = a_5 = a_7 = 1$ and plotting the period as a function of the energy. It can be appreciated that our method (solid line) provides an excellent approximation to the exact solution (dotted line) even for oscillations of large amplitude, whereas the LP expansion (dashed line) only works for very small oscillations.

6. The nonlinear pendulum

We now apply the improved method to the nonlinear pendulum. The steps are exactly the same as before and we proceed to outline them. First, consider the equation for the nonlinear pendulum,

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0, \tag{35}$$

where $\omega^2 = g/l$ is the natural frequency of the small oscillations of the pendulum. Following the Lindstedt–Poincaré method, we introduce a scaled time $\tau = \Omega t$ and write the equation as

$$\Omega^2 \frac{d^2\theta}{d\tau^2} + \omega^2 \sin\theta = 0, \tag{36}$$

where $\Omega = 2\pi/T$ is the (unknown) frequency and T is the period of the oscillations. As discussed in the case of the anharmonic oscillator, we can apply the Linear Delta Expansion to the problem by modifying the above equation and writing it as

$$\Omega^2 \frac{d^2\theta}{d\tau^2} + \lambda^2\theta = \delta[-\omega^2 \sin\theta + \lambda^2\theta] \equiv \delta f(\theta), \tag{37}$$

where λ is an arbitrary parameter, with the dimension of frequency. In what follows we use the same procedure previously outlined for the anharmonic oscillator, with a few technical differences due to the more difficult nature of the present problem.

We will expand the angle and the frequency as

$$\theta(\tau) = \sum_{n=0}^{\infty} \delta^n \theta_n(\tau), \quad \Omega^2 = \sum_{n=0}^{\infty} \delta^n \alpha_n.$$

We will solve Eq. (37) subject to the boundary condition $\theta(0) = A$ and $\dot{\theta}(0) = 0$, i.e.

$$\theta_0(0) = A, \quad \theta_{j>0}(0) = 0, \quad \dot{\theta}_j = 0. \tag{38}$$

6.1. Zeroth order

To zeroth order the equation for the pendulum reads

$$\alpha_0 \frac{d^2\theta_0}{d\tau^2} + \lambda^2\theta_0 = 0, \tag{39}$$

and we obtain the solution

$$\theta_0(\tau) = A\cos\tau, \tag{40}$$

describing a simple oscillatory motion with (scaled) period 2π. The zeroth-order frequency is therefore given by

$$\alpha_0 = \lambda^2. \tag{41}$$

6.2. First order

To first order we obtain the differential equation

$$\alpha_0 \frac{d^2\theta_1}{d\tau^2} + \lambda^2\theta_1 = S_1(\tau), \tag{42}$$

where we have defined the source term:

$$S_1(\tau) \equiv -\alpha_1 \frac{d^2\theta_0}{d\tau^2} + f(\theta_0) = A\alpha_1 \cos\tau + [-\omega^2 \sin(A\cos\tau) + \lambda^2 A\cos\tau]. \tag{43}$$

As before, in order to avoid the occurrence of secular terms, we need to eliminate contributions proportional to $\cos\tau$ from the source term $S_1(\tau)$ (recall that such a term would yield a resonant behavior of the solution $\theta_1(\tau)$). We enforce this condition by requiring that

$$\frac{1}{\pi}\int_0^{2\pi} d\tau S_1(\tau) e^{i\tau} = 0. \tag{44}$$

As a result of this operation, we are able to fix the coefficient α_1:

$$\alpha_1 = \frac{1}{A}\left[-\lambda^2 A + \frac{\omega^2}{\pi}\int_0^{2\pi} d\tau \sin(A\cos\tau) e^{i\tau}\right] = \frac{1}{A}[-\lambda^2 A + \omega^2 c_1], \tag{45}$$

where we have used the following expansion of $\sin(A\cos\tau)$:

$$\sin(A\cos\tau) = \sum_{j=0}^{\infty} c_{2j+1}\cos[(2j+1)\tau], \tag{46}$$

and

$$c_{2j+1} = \frac{1}{\pi}\int_0^{2\pi} d\tau e^{i(2j+1)\tau} \sin(A\cos\tau) = 2(-1)^j J_{2j+1}(A). \tag{47}$$

Eq. (42) now reads

$$\alpha_0 \frac{d^2\theta_1}{d\tau^2} + \lambda^2\theta_1 = S_1(\tau) = -\omega^2 \sum_{j=1}^{\infty} c_{2j+1}\cos[(2j+1)\tau], \tag{48}$$

where the sum starts from $j = 1$ because of the vanishing of the term proportional to $\cos\tau$.

We write the solution $\theta_1(\tau)$ as

$$\theta_1(\tau) = \sum_{j=0}^{\infty} d_{2j+1}^{(1)} \cos[(2j+1)\tau], \tag{49}$$

where the coefficients are (for $j > 1$)

$$d_{2j+1}^{(1)} = \frac{\omega^2 c_{2j+1}}{4\lambda^2 j(j+1)} \equiv \frac{\overline{d}_{2j+1}^{(1)}}{\lambda^2}. \tag{50}$$

In the last equation we have introduced the scale coefficients $\overline{d}_{2j+1}^{(1)}$, which do not depend upon λ. We notice that Eq. (48) cannot be used to determine the coefficient corresponding to $j = 0$; in fact, this coefficient is fixed by the boundary condition

$$\theta_1(0) = \sum_{j=0}^{\infty} d_{2j+1}^{(1)} = 0, \tag{51}$$

which entails

$$d_1^{(1)} = -\sum_{j=1}^{\infty} d_{2j+1}^{(1)} = -\sum_{j=1}^{\infty} \frac{\omega^2 c_{2j+1}}{4\lambda^2 j(j+1)} \equiv \frac{\overline{d}_1^{(1)}}{\lambda^2}. \tag{52}$$

6.3. Second order

To second order we obtain the equation

$$\alpha_0 \frac{\mathrm{d}^2\theta_2}{\mathrm{d}\tau^2} + \lambda^2\theta_2 = S_2(\tau), \tag{53}$$

where we have introduced the source term

$$\begin{aligned} S_2(\tau) \equiv & -\alpha_1 \frac{\mathrm{d}^2\theta_1}{\mathrm{d}\tau^2} - \alpha_2 \frac{\mathrm{d}^2\theta_0}{\mathrm{d}\tau^2} + \theta_1(\tau) f'(\theta_0) \\ = & -\alpha_1 \frac{\mathrm{d}^2\theta_1}{\mathrm{d}\tau^2} - \alpha_2 \frac{\mathrm{d}^2\theta_0}{\mathrm{d}\tau^2} + \theta_1(\tau)[-\omega^2 \cos\theta_0(\tau) + \lambda^2]. \end{aligned} \tag{54}$$

We can expand the source term in a series as

$$S_2(\tau) = \sum_{n=1}^{\infty} s_{2n+1}^{(2)} \cos(2n+1)\tau, \tag{55}$$

where the coefficients of the expansion are given by

$$s_{2n+1}^{(2)} = \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\tau \mathrm{e}^{\mathrm{i}(2n+1)\tau} S_2(\tau) \equiv \frac{\overline{s}_{2n+1}^{(2a)}}{\lambda^2} + \overline{s}_{2n+1}^{(2b)}. \tag{56}$$

We have introduced the scaled coefficients $\overline{s}_{2n+1}^{(2a)}$ and $\overline{s}_{2n+1}^{(2b)}$, which are independent of λ and read

$$\begin{aligned} \overline{s}_{2n+1}^{(2a)} = & \frac{\omega^2 c_1}{A}(2n+1)^2 \overline{d}_{2n+1}^{(1)} \\ & - \frac{\omega^2}{2}\left\{ \sum_{j=n}^{\infty} \overline{d}_{2j+1}^{(1)} \tilde{c}_{2(j-n)} + \sum_{l=n+1}^{\infty} \overline{d}_{2(l-n-1)+1}^{(1)} \tilde{c}_{2l} + \sum_{j=0}^{n} \overline{d}_{2j+1}^{(1)} \tilde{c}_{2(n-j)} \right\}, \end{aligned}$$

$$\overline{s}_{2n+1}^{(2b)} = -4n(n+1)\overline{d}_{2n+1}^{(1)}.$$

The coefficients $\tilde{c}_{2j}$ follow from the expansion of $\cos[A\cos\tau]$:

$$\cos[A\cos\tau] = \sum_{j=0}^{\infty} \tilde{c}_{2j}\cos[2j\tau], \tag{57}$$

and read, for $j>0$,

$$\begin{aligned}\tilde{c}_{2j} &\equiv \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\tau \cos[A\cos\tau]\mathrm{e}^{\mathrm{i}2j\tau} \\ &= 2\sum_{n=j}^{\infty}(-1)^n\left(\frac{A}{2}\right)^{2n}\frac{1}{(n-j)!(n+j)!} = 2(-1)^j J_{2j}(A)\end{aligned} \tag{58}$$

and, for $j=0$,

$$\tilde{c}_0 = \cos A - \sum_{j=1}^{\infty}\tilde{c}_{2j}. \tag{59}$$

As before we need to eliminate the coefficient $s_1^{(2)}$:

$$s_1^{(2)} = \alpha_1 d_1^{(1)} + \alpha_2 A + \lambda^2 d_1^{(1)} - \frac{\omega^2}{2}\sum_{j=0}^{\infty} d_{2j+1}^{(1)}(\tilde{c}_{2j}+\tilde{c}_{2j+2}) - \frac{\omega^2}{2}d_1^{(1)}\tilde{c}_0 \tag{60}$$

and obtain the coefficient α_2:

$$\alpha_2 = \frac{1}{A}\left\{-\left(\frac{\omega^2 c_1}{A} - \frac{\omega^2\tilde{c}_0}{2}\right)\frac{\overline{d}_1^{(1)}}{\lambda^2} + \frac{\omega^2}{2\lambda^2}\sum_{j=0}^{\infty}\overline{d}_{2j+1}^{(1)}(\tilde{c}_{2j}+\tilde{c}_{2j+2})\right\} \equiv \frac{\overline{\alpha}_2}{\lambda^2}, \tag{61}$$

where $\overline{\alpha}_2 = \lambda^2\alpha_2$ is a scaled coefficient, independent of λ.

We are therefore able to find the solution of Eq. (53),

$$\theta_2(\tau) = \sum_{j=0}^{\infty} d_{2j+1}^{(2)}\cos[(2j+1)\tau], \tag{62}$$

with the coefficients, for $j \neq 1$,

$$d_{2j+1}^{(2)} \equiv \frac{\overline{d}_{2j+1}^{(2a)}}{\lambda^4} + \frac{\overline{d}_{2j+1}^{(2b)}}{\lambda^2}, \tag{63}$$

expressed in terms of the λ-independent terms

$$\overline{d}_{2j+1}^{(2a)} = -\frac{\overline{s}_{2j+1}^{(2a)}}{4j(j+1)}, \quad \overline{d}_{2j+1}^{(2b)} = -\frac{\overline{s}_{2j+1}^{(2b)}}{4j(j+1)} = \overline{d}_{2j+1}^{(1)}.$$

As before the $j=0$ coefficient is not fixed by the equation and needs to be determined by enforcing the boundary condition $\theta_2(0)=0$. We obtain

$$d_1^{(2)} = -\sum_{j=1}^{\infty} d_{2j+1}^{(2)} = \sum_{j=1}^{\infty} \frac{s_{2j+1}^{(2)}}{4\lambda^2 j(j+1)}. \tag{64}$$

6.4. Third order

To third order we obtain the equation

$$\alpha_0 \frac{d^2\theta_3}{d\tau^2} + \lambda^2\theta_3 = S_3(\tau), \tag{65}$$

where the source term $S_3(\tau)$ is

$$S_3(\tau) \equiv -\alpha_1 \frac{d^2\theta_2}{d\tau^2} - \alpha_2 \frac{d^2\theta_1}{d\tau^2} - \alpha_3 \frac{d^2\theta_0}{d\tau^2} + \left[\theta_2(\tau) f'(\theta_0) + \frac{\theta_1^2(\tau)}{2} f''(\theta_0)\right]. \tag{66}$$

Once again it is useful to expand the source term in a series as

$$S_3(\tau) = \sum_{n=0}^{\infty} s_{2n+1}^{(3)} \cos(2n+1)\tau, \tag{67}$$

where the coefficients of the expansion are given by

$$s_{2n+1}^{(3)} = \frac{1}{\pi}\int_0^{2\pi} d\tau e^{i(2n+1)\tau} S_3(\tau) = \frac{\bar{s}_{2n+1}^{(3a)}}{\lambda^4} + \frac{\bar{s}_{2n+1}^{(3b)}}{\lambda^2} + \bar{s}_{2n+1}^{(3c)} \tag{68}$$

and $\bar{s}^{(3a,b,c)}$ are independent of λ. A lengthy calculation allows to find the expressions for these coefficients, which can be found in Appendix A. Here we only write the coefficient of the term $\cos\tau$, corresponding to $n=0$:

$$\begin{aligned} s_1^{(3)} &= \alpha_1 d_1^{(2)} + \alpha_2 d_1^{(1)} + \alpha_3 A - \frac{\omega^2}{2}\sum_{j=0}^{\infty} \tilde{c}_{2j} d_{2j+1}^{(2)} - \frac{\omega^2}{2}\sum_{l=1}^{\infty} \tilde{c}_{2l} d_{2l-1}^{(2)} - \frac{\omega^2}{2}\tilde{c}_0 d_1^{(2)} + \lambda^2 d_1^{(2)} \\ &+ \frac{\omega^2}{8}\sum_{m=0}^{\infty}\sum_{j=m+1}^{\infty} c_{2(j-m-1)+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} + \frac{\omega^2}{8}\sum_{j=0}^{\infty}\sum_{m=0}^{j} c_{2(-m+j)+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} \\ &+ \frac{\omega^2}{8}\sum_{m=j+1}^{\infty}\sum_{j=0}^{\infty} c_{2(m-j-1)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)} + \frac{\omega^2}{8}\sum_{m=0}^{\infty}\sum_{j=0}^{m} c_{2(m-j)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)} \\ &+ \frac{\omega^2}{8}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} c_{2(m+j)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)} + \frac{\omega^2}{8}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} c_{2(m+j+1)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)}. \end{aligned} \tag{69}$$

The coefficient α_3 is fixed by requiring that $s_1^{(3)}$ vanish:

$$\alpha_3 = \frac{\alpha_{3a}}{\lambda^4} + \frac{\alpha_{3b}}{\lambda^2}, \tag{70}$$

where

$$\alpha_{3a} = -\frac{\omega^2}{A}\left\{\left(\frac{c_1}{A}-\frac{\tilde{c}_0}{2}\right)\overline{d}_1^{(2a)}+\overline{\alpha}_2\overline{d}_1^{(1)}-\frac{1}{2}\sum_{j=0}^{\infty}(\tilde{c}_{2j}+\tilde{c}_{2j+2})\overline{d}_{2j+1}^{(2a)}\right.$$
$$+\frac{1}{8}\sum_{m=0}^{\infty}\left[\sum_{j=m+1}^{\infty}(2c_{2(j-m-1)+1}+c_{2(m+j)+1}+c_{2(m+j)+3})\overline{d}_{2j+1}^{(1)}\overline{d}_{2m+1}^{(1)}\right.$$
$$\left.\left.+\sum_{j=0}^{m}(2c_{2(m-j)+1}+c_{2(m+j)+1}+c_{2(m+j)+3})\overline{d}_{2j+1}^{(1)}\overline{d}_{2m+1}^{(1)}\right]\right\},$$
$$\alpha_{3b} = -\frac{\omega^2}{A}\left\{\left(\frac{c_1}{A}-\frac{\tilde{c}_0}{2}\right)\overline{d}_1^{(2b)}-\frac{1}{2}\sum_{j=0}^{\infty}(\tilde{c}_{2j}+\tilde{c}_{2j+2})\overline{d}_{2j+1}^{(2b)}\right\}=\overline{\alpha}_2. \tag{71}$$

To this order the squared frequency reads

$$\Omega^2 = \alpha_{1a} + 2\frac{\overline{\alpha}_2}{\lambda^2}+\frac{\alpha_{3a}}{\lambda^4}.$$

The "principle of minimal sensitivity" yields the solution

$$\lambda^2 = -\frac{\alpha_{3a}}{\overline{\alpha}_2} \tag{72}$$

and a corresponding value of Ω^2:

$$\Omega^2 = \alpha_{1a} - \frac{\overline{\alpha}_2^2}{\alpha_{3a}}. \tag{73}$$

In Fig. 7 we plot the period of the nonlinear pendulum as a function of the amplitude, as obtained in the LPLDE and LP approximations, and compare the results with the exact period.

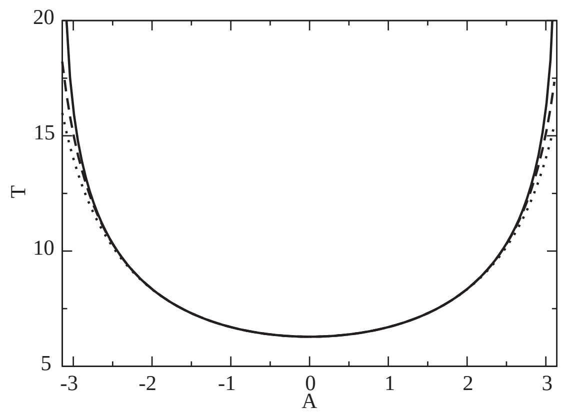

Fig. 7. Period of the nonlinear pendulum as a function of the amplitude. We assume $\omega = \sqrt{g/l} = 1$. The solid line is the exact result, the dashed line is the result obtained with our method and the dotted line is the Lindstedt–Poincaré result.

We assume $\omega = 1$ and use the formulae given above truncating the infinite series to a maximum value $j_{\max} = 5$. As it can be seen from the figure, the LPLDE approximation is in excellent agreement with the exact result, up to very large amplitudes. $A = \pm\pi$ corresponds to an unstable point of equilibrium, for which the exact period diverges.

7. Conclusions

We have presented a method for the solution of nonlinear problems which are conservative and periodic. It is based on the application of the Linear Delta Expansion to the Lindstedt–Poincaré method. We applied it to three problems: the Duffing Equation, more general anharmonic potentials and the nonlinear pendulum. In the case of the Duffing equation we find that the new method converges faster and with greater accuracy than the simple LP method. Also, by comparing it with methods based on the perturbative δ expansion, we show that our solution not only converges faster and more accurately, but it also works for a much wider range of parameters, including the case in which the nonlinear coupling μ is negative. In a similar fashion, we show that the method works remarkably well also for the nonlinear pendulum, for which the method is implemented without performing any Taylor expansion of the potential. Recently we have also obtained an extension of the present method to quantum systems, which is based on the ideas of multiple scale analysis [13].

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Appendix A. Coefficients

In this appendix we present the computation of the coefficients of $s^{(3)}_{2n+1}$ in Eq. (68). Let us rewrite Eq. (68) in the following form:

$$s^{(3)}_{2n+1} \equiv I^{(A)}_{2n+1} + I^{(B)}_{2n+1} + I^{(C)}_{2n+1} + I^{(D)}_{2n+1}. \tag{A.1}$$

We now proceed to compute each of these terms:

- $I^{(A)}_{2n+1}$:

$$I^{(A)}_{2n+1} = \left(\frac{\omega^2 c_1}{A} - \lambda^2\right)(2n+1)^2\left[\frac{\overline{d}^{(2a)}_{2n+1}}{\lambda^4} + \frac{\overline{d}^{(2b)}_{2n+1}}{\lambda^2}\right] + \frac{\overline{\alpha}_2}{\lambda^2}(2n+1)^2\frac{\overline{d}^{(1)}_{2n+1}}{\lambda^2} + \alpha_3 A\delta_{n0} \tag{A.2}$$

$$\equiv \frac{i^{(1)}_a}{\lambda^4} + \frac{i^{(2)}_a}{\lambda^2} + i^{(3)}_a. \tag{A.3}$$

- $I^{(B)}_{2n+1}$:

$$I^{(B)}_{2n+1} = -\frac{\omega^2}{2}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\tilde{c}_{2l}d^{(2)}_{2j+1}\left[\delta_{n+l-j,0}+\delta_{n+1-l+j,0}+\delta_{l+j-n,0}+\delta_{n+l+j+1,0}\right]$$
$$\equiv \frac{i_b^{(1)}}{\lambda^4}+\frac{i_b^{(2)}}{\lambda^2}. \tag{A.4}$$

The four different integrals become:

(a)
$$I^{(B1)}_{2n+1} = -\frac{\omega^2}{2}\sum_{j=n}^{\infty}\tilde{c}_{2(j-n)}d^{(2)}_{2j+1}, \tag{A.5}$$

(b)
$$I^{(B2)}_{2n+1} = -\frac{\omega^2}{2}\sum_{l=n+1}^{\infty}\tilde{c}_{2l}d^{(2)}_{2(l-n-1)+1}, \tag{A.6}$$

(c)
$$I^{(B3)}_{2n+1} = -\frac{\omega^2}{2}\sum_{l=0}^{n}\tilde{c}_{2l}d^{(2)}_{2(n-l)+1}, \tag{A.7}$$

(d)
$$I^{(B4)}_{2n+1} = 0. \tag{A.8}$$

Therefore, we finally have that

$$I^{(B)}_{2n+1} = -\frac{\omega^2}{2}\left[\sum_{j=n}^{\infty}\tilde{c}_{2(j-n)}d^{(2)}_{2j+1}+\sum_{l=n+1}^{\infty}\tilde{c}_{2l}d^{(2)}_{2(l-n-1)+1}+\sum_{l=0}^{n}\tilde{c}_{2l}d^{(2)}_{2(n-l)+1}\right], \tag{A.9}$$

- $I^{(C)}_{2n+1}$:

$$I^{(C)}_{2n+1} = \frac{1}{\pi}\int_0^{2\pi} d\tau e^{i(2n+1)\tau}\lambda^2\left(\sum_{j=0}^{\infty}d^{(2)}_{2j+1}\cos(2j+1)\tau\right) = \lambda^2 d^{(2)}_{2n+1}$$
$$\equiv \frac{i_c^{(1)}}{\lambda^2}+i_c^{(2)}, \tag{A.10}$$

- $I^{(D)}_{2n+1}$:

$$I^{(D)}_{2n+1} = \frac{1}{\pi}\int_0^{2\pi} d\tau e^{i(2n+1)\tau}\frac{\omega^2}{2}\sin A\cos\tau\left(\sum_{j=0}^{\infty}d^{(1)}_{2j+1}\cos(2j+1)\tau\right)^2$$

$$= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\tau \mathrm{e}^{\mathrm{i}(2n+1)\tau} \frac{\omega^2}{2}\sum_{l=0}^{\infty} c_{2l+1}\cos(2l+1)\tau \times \sum_{m=0}^{\infty} d_{2m+1}^{(1)}\cos(2m+1)\tau \sum_{j=0}^{\infty} d_{2j+1}^{(1)}\cos(2j+1)\tau.$$

We need to calculate the following integral:

$$\mathscr{I} = \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\tau \mathrm{e}^{\mathrm{i}(2n+1)\tau}\cos[(2j+1)\tau]\cos[(2l+1)\tau]\cos[(2m+1)\tau]. \tag{A.12}$$

Using the relation

$$\begin{aligned}\mathscr{C} =& \cos[(2j+1)\tau]\cos[(2l+1)\tau]\cos[(2m+1)\tau]\\ =& \tfrac{1}{4}[\cos(2(l+m+j)+3)\tau + \cos(2(l+m-j)+1)\tau\\ &+ \cos(2(l-m+j)+1)\tau + \cos(2(l-m-j)-1)\tau],\end{aligned} \tag{A.13}$$

one obtains

$$\begin{aligned}\mathscr{I} =& \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\tau \mathrm{e}^{\mathrm{i}(2n+1)\tau}\mathscr{C}\\ =& \tfrac{1}{4}\{\delta_{2(n+l+m+j)+4,0} + \delta_{2(n-l-m-j)-2,0} + \delta_{2(n+l+m-j)+2,0} + \delta_{2(n-l-m+j),0}\\ &+ \delta_{2(n+l-m+j)+2,0} + \delta_{2(n-l+m-j),0} + \delta_{2(n+l-m-j),0} + \delta_{2(n-l+m+j)+2,0}\},\end{aligned} \tag{A.14}$$

and finally

$$\begin{aligned}I_{2n+1}^{(D)} =& \frac{\omega^2}{8}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} c_{2l+1}d_{2m+1}^{(1)}d_{2j+1}^{(1)}\{\delta_{2(n+l+m+j)+4,0} + \delta_{2(n-l-m-j)-2,0} + \delta_{2(n+l+m-j)+2,0}\\ &+\delta_{2(n-l-m+j),0} + \delta_{2(n+l-m+j)+2,0} + \delta_{2(n-l+m-j),0} + \delta_{2(n+l-m-j),0} + \delta_{2(n-l+m+j)+2,0}\}\\ =& \frac{i_d}{\lambda^4}.\end{aligned} \tag{A.15}$$

We are then left with 8 integrals that can be evaluated in the following way (we call them $I_D^{(i)}$):

(i)

$$I_D^{(1)} = \frac{\omega^2}{8}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} c_{2l+1}d_{2m+1}^{(1)}d_{2j+1}^{(1)}\delta_{2(n+l+m+j)+4,0} = 0, \tag{A.16}$$

(ii)

$$\begin{aligned}I_D^{(2)} =& \frac{\omega^2}{8}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} c_{2l+1}d_{2m+1}^{(1)}d_{2j+1}^{(1)}\delta_{2(n-l-m-j)-2,0}\\ =& \frac{\omega^2}{8}\sum_{m=0}^{n-j-1}\sum_{j=0}^{n-1} c_{2(n-m-j-1)+1}d_{2m+1}^{(1)}d_{2j+1}^{(1)},\end{aligned} \tag{A.17}$$

(iii)

$$I_D^{(3)} = \frac{\omega^2}{8} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{2l+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} \delta_{2(n+l+m-j)+2,0}$$
$$= \frac{\omega^2}{8} \sum_{m=0}^{\infty} \sum_{j=n+m+1}^{\infty} c_{2(j-n-m-1)+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)}, \qquad \text{(A.18)}$$

(iv)

$$I_D^{(4)} = \frac{\omega^2}{8} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{2l+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} \delta_{2(n-l-m+j),0}$$
$$= \frac{\omega^2}{8} \sum_{j=0}^{\infty} \sum_{m=0}^{n+j} c_{2(n-m+j)+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)}, \qquad \text{(A.19)}$$

(v)

$$I_D^{(5)} = \frac{\omega^2}{8} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{2l+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} \delta_{2(n+l-m+j)+2,0}$$
$$= \frac{\omega^2}{8} \sum_{m=n+j+1}^{\infty} \sum_{j=0}^{\infty} c_{2(m-n-j-1)+1} d_{2j+1}^{(1)} D_{2cm+1}^{(1)}, \qquad \text{(A.20)}$$

(vi)

$$I_D^{(6)} = \frac{\omega^2}{8} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{2l+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} \delta_{2(n-l+m-j),0}$$
$$= \frac{\omega^2}{8} \sum_{m=0}^{\infty} \sum_{j=0}^{m+n} c_{2(m+n-j)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)}, \qquad \text{(A.21)}$$

(vii)

$$I_D^{(7)} = \frac{\omega^2}{8} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{2l+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} \delta_{2(n+l-m-j),0}$$
$$= \frac{\omega^2}{8} \sum_{m=\max(0,n-j)}^{\infty} \sum_{j=0}^{\infty} c_{2(m+j-n)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)}, \qquad \text{(A.22)}$$

(viii)

$$I_D^{(8)} = \frac{\omega^2}{8} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{2l+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} \delta_{2(n-l+m+j)+2,0}$$
$$= \frac{\omega^2}{8} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{2(n+m+j+1)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)}. \qquad \text{(A.23)}$$

The final expression is

$$
\begin{aligned}
I_{2n+1}^{(D)} = \frac{\omega^2}{8}\Bigg\{ & \sum_{m=0}^{n-j-1}\sum_{j=0}^{n-1} c_{2(n-m-j-1)+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} + \sum_{m=0}^{\infty}\sum_{j=n+m+1}^{\infty} c_{2(j-n-m-1)+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} \\
& + \sum_{j=0}^{\infty}\sum_{m=0}^{n+j} c_{2(n-m+j)+1} d_{2m+1}^{(1)} d_{2j+1}^{(1)} + \sum_{m=n+j+1}^{\infty}\sum_{j=0}^{\infty} c_{2(m-n-j-1)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)} \\
& + \sum_{m=0}^{\infty}\sum_{j=0}^{m+n} c_{2(m+n-j)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)} + \sum_{m=\max(0,n-j)}^{\infty}\sum_{j=0}^{\infty} c_{2(m+j-n)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)} \\
& + \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} c_{2(n+m+j+1)+1} d_{2j+1}^{(1)} d_{2m+1}^{(1)} \Bigg\}. \qquad \text{(A.24)}
\end{aligned}
$$

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